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A FINITE ELEMENT METHOD FOR TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we consider the finite element method for time fractional partial differential equations. The existence and uniqueness of the solutions are proved by using the Lax-Milgram Lemma. A time stepping method is introduced based on a quadrature formula approach. The fully discrete scheme is considered by using a finite element method and optimal convergence error estimates are obtained. The numerical examples at the end of the paper show that the experimental results are consistent with our theoretical results.

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Key Words and Phrases: Fractional partial differential equations, finite element method, error estimates, numerical examples

1. Introduction

In this paper, we will consider the finite element method for the time fractional partial differential equation

$$ {}_0^R D_t^{\alpha} u(t,x) - \Delta u(t,x) = f(t,x), \quad t \in [0,T],\ x \in \Omega, \tag{1.1} $$

$$ u(0,x) = 0, \quad x \in \Omega, \tag{1.2} $$

$$ u(t,x) = 0, \quad t \in [0,T],\ x \in \partial\Omega, \tag{1.3} $$

where $0 < \alpha < 1$ and Ω is the bounded open domain in $\mathbb{R}^d, d = 1,2,3$ and $\partial\Omega$ is the boundary of Ω. Here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ denotes the Laplacian operator with respect to the x variable, ${}_0^R D_t^{\alpha} u(t,x)$ denotes the left

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Riemann-Liouville fractional derivative with respect to the time variable t defined by

$$ {}_0^R D_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(\tau,x)}{(t-\tau)^\alpha}\, d\tau, \quad 0<\alpha<1, \tag{1.4} $$

where Γ denotes the Gamma function.

Time fractional partial differential equations have many applications in areas such as diffusion processes, electromagnetics, electrochemistry, material science, turbulent flow, chaotic dynamics, etc. [3], [4], [14], [15], [24], [25], [28], [29]. Analytical solutions of time fractional partial differential equations have been studied using Green's functions or Fourier-Laplace transforms [26], [23], [30], [31].

Numerical methods for fractional ordinary differential equations were studied in, for example, [6], [7], [8], [9] [12], [13]. Numerical methods for fractional partial differential equations were also studied by some authors. Liu et al. [22] employed the finite difference method in both space and time and analyzed the stability condition. Sun and Wu [32] proposed a finite difference method for the fractional diffusion-wave equation. Langlands and Henry [18] considered an implicit numerical scheme for fractional diffusion equation. Lin and Xu [20] proposed a finite difference method in time and Legendre spectral method in space. Li and Xu [19] proposed a time-space spectral method for time-space fractional partial differential equation based on a weak formulation and a detailed error analysis was carried out.

Recently, Ervin and Roop [10], [11] used finite element methods to find the variational solution of the fractional advection dispersion equation, in which the fractional derivative depends on the space, related to the nonlocal operator, but the time derivative term is of first order, related to the local operator. Adolfsson et al. [1], [2] considered an efficient numerical method to integrate the constitutive response of fractional order viscoelasticity based on the finite element method. Li et al. [16] considered a time fractional partial differential equation by using the finite element method and obtained error estimates in both semidiscrete and fully discrete cases. Jiang et al. [17] considered a high-order finite element method for the time fractional partial differential equaions and proved the optimal order error estimates.

In this paper, we will use the framework in Li and Xu [19] in which the authors introduced suitable spaces and norms in which the time fractional differential problem can be formulated into an elliptic problem. Using these spaces, we introduce a finite element method for time fractional partial differential equation and obtain the optimal order error estimates both in semidiscrete and fully discrete cases.

The paper is organized as follows. In Section 2, we consider the existence and uniqueness of the solution of the time fractional partial differential equation. In Section 3, we introduce a time discretization scheme and prove the error estimate. In Section 4, we consider the finite element method and obtain the optimal order error estimates in space discretization. Finally in Section 5, we give two numerical examples and show that the numerical results are consistent with the theoretical results.

2. Existence and uniqueness

Let $C^\infty(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$ and $C_0^\infty(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$ with compact support in $(0,T)$. Let ${}_0C^\infty(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$ with compact support in $(0,T]$. Then we introduce the following Sobolev space ${}_0H^\alpha(0,T),\ 0<\alpha<1$ which is the closure of ${}_0C^\infty(0,T)$ with respect to the norm $\|\cdot\|_{H^\alpha(0,T)}$, where $\|\cdot\|_{H^\alpha(0,T)}$ denotes the norm in the usual fractional Sobolev space $H^\alpha(0,T)$ [21]. Further, let $L_2(\Omega), H^1(\Omega), H^2(\Omega)$ denote the usual Sobolev spaces with corresponding norms $\|\cdot\|_{L_2(\Omega)}, \|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{H^2(\Omega)}$, respectively. Denote $H_0^1(\Omega)=\{v\in H^1(\Omega),\ v|_{\partial\Omega}=0\}$ with norm $\|\cdot\|_{H^1(\Omega)}$.

Define the space, with $0<\alpha<1$,

$$B^\alpha\big((0,T)\times\Omega\big) = {}_0H^\alpha\big((0,T),L_2(\Omega)\big)\cap L_2\big((0,T),H_0^1(\Omega)\big).$$

Here $B^\alpha\big((0,T)\times\Omega\big)$ is a Banach space with respect to the norm

$$\|v\|_{((0,T)\times\Omega)} = \big(\|v\|_{H^\alpha((0,T),L_2(\Omega))} + \|v\|_{L_2((0,T),H_0^1(\Omega))}\big)^{1/2}.$$

We have the following existence and uniqueness theorem.

THEOREM **2.1**. *Assume that $0<\alpha<1$ and $f\in L_2((0,T)\times\Omega)$. Then the system* (1.1) - (1.3) *has a unique solution in $B^\alpha\big((0,T)\times\Omega\big)$. Further the following stability result holds:*

$$\|u\|_{B^{\alpha/2}((0,T)\times\Omega} \le C\|f\|_{L_2((0,T)\times\Omega)}. \tag{2.1}$$

The proof of Theorem **2.1** can be found in [19]. For completeness, and because we use the approach later, we will give the ideas of the proof of this theorem here.

Recall that the right Riemann-Liouville fractional integral is defined as

$${}_t^R D_T^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{v(\tau)}{(t-\tau)^\alpha}\,d\tau, \quad 0<\alpha<1. \tag{2.2}$$

Definition **2.1**. Define $H_r^\alpha(0,T)$ as the closure of $C_0^\infty(0,T)$ with respect to the norm $\|\cdot\|_{H_r^\alpha(0,T)}$, that is,

$$H_r^\alpha(0,T) = \Big\{ v \in L_2(0,T) \Big| \, \exists\, v_n \in C_0^\infty(0,T), \text{ such that } \|v_n - v\|_{H_r^\alpha(0,T)} \to 0 \Big\}.$$

Here the norm $\|\cdot\|_{H_r^\alpha(0,T)}$ is defined by

$$\|v\|_{H_r^\alpha(0,T)} = \Big(\|v\|^2_{L_2(0,T)} + |v|^2_{H_r^\alpha(0,T)} \Big)^{1/2},$$

where the seminorm $|\cdot|_{H_r^\alpha(0,T)}$ is defined by

$$|v|_{H_r^\alpha(0,T)} = \|{}_t^R D_T^\alpha v\|_{L_2(0,T)}.$$

Definition **2.2**. Define $H_c^\alpha(0,T)$ as the closure of $C_0^\infty(0,T)$ with respect to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$, that is,

$$H_c^\alpha(0,T) = \Big\{ v \in L_2(0,T) \Big| \, \exists\, v_n \in C_0^\infty(0,T), \text{ such that } \|v_n - v\|_{H_c^\alpha(0,T)} \to 0 \Big\}.$$

Here the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ is defined by

$$\|v\|_{H_c^\alpha(0,T)} = \Big(\|v\|^2_{L_2(0,T)} + |v|^2_{H_c^\alpha(0,T)} \Big)^{1/2},$$

where the seminorm $|\cdot|_{H_c^\alpha(0,T)}$ is defined by

$$|v|_{H_c^\alpha(0,T)} = \big| \big({}_0D_t^\alpha v, \, {}_t^R D_T^\alpha v \big)_{L_2(0,T)} \big|^{1/2},$$

where $(\cdot,\cdot)_{L_2(0,T)}$ denotes the inner product in $L_2(0,T)$.

Lemma **2.1**. *Let $0 < \alpha < 1$. We have*

$$H_r^\alpha(0,T) = H_c^\alpha(0,T) = H_0^\alpha(0,T),$$

and the norms $\|\cdot\|_{H_r^\alpha(0,T)}, \|\cdot\|_{H_c^\alpha(0,T)}$ and $\|\cdot\|_{H_0^\alpha(0,T)}$ are equivalent.

P r o o f. We first prove $H_r^\alpha(0,T) = H_c^\alpha(0,T)$. In fact, $\forall\, v \in H_r^\alpha(0,T)$, there exists a sequence $v_n \in C_0^\infty(0,T)$ such that

$$\|v_n - v\|_{H_r^\alpha(0,T)} \to 0, \quad n \to \infty,$$

which implies that $\|v_n - v_m\|_{H_r^\alpha(0,T)} \to 0, \ n \to \infty, \ m \to \infty$, that is, $\{v_n\}$ is a Cauchy sequence in $H_r^\alpha(0,T)$. We will show that the norm $\|\cdot\|_{H_r^\alpha(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ in $C_0^\infty(0,T)$. Assuming this for the moment, we see that $\{v_n\}$ is also a Cauchy sequence in $\|\cdot\|_{H_c^\alpha(0,T)}$. Thus there exists $v' \in H_c^\alpha(0,T)$ such that

$$\|v_n - v'\|_{H_c^\alpha(0,T)} \to 0, \quad n \to \infty.$$

Hence, we have

$$\begin{aligned}\|v-v'\|_{L_2(0,T)} &\le \|v-v_n\|_{L_2(0,T)}+\|v_n-v'\|_{L_2(0,T)}\\ &\le \|v-v_n\|_{H_r^\alpha(0,T)}+\|v_n-v'\|_{H_c^\alpha(0,T)}\to 0,\quad n\to\infty,\end{aligned} \tag{2.3}$$

which implies that $v=v'$ and therefore $H_r^\alpha(0,T)\subset H_c^\alpha(0,T)$. Similarly, we can prove $H_c^\alpha(0,T)\subset H_r^\alpha(0,T)$. Hence we get

$$H_c^\alpha(0,T)=H_r^\alpha(0,T).$$

Denote the norm equivalence by $\|\cdot\|_{H_r^\alpha(0,T)}\cong\|\cdot\|_{H_c^\alpha(0,T)}$. We now need to prove that the norm $\|\cdot\|_{H_r^\alpha(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ in $C_0^\infty(0,T)$. In fact, $\forall\, v\in C_0^\infty(0,T)$, let $\tilde v$ be the extension of v by zero outside $(0,T)$. Then we have

$$|v|_{H_r^\alpha(0,T)}=|\tilde v|_{H_r^\alpha(\mathbf{R})}\cong|\tilde v|_{H_c^\alpha(\mathbf{R})}=|v|_{H_c^\alpha(0,T)},$$

where we use the fact that

$$|v|_{H_c^\alpha(0,T)}=|\tilde v|_{H_c^\alpha(\mathbf{R})},$$

which follows from

$$|\tilde v|_{H_c^\alpha(\mathbf{R})}=\left|\left({}_{-\infty}D_t^\alpha\tilde v,\,{}_tD_{+\infty}^\alpha\tilde v\right)\right|^{1/2}=\left|\left({}_0D_t^\alpha\tilde v,\,{}_tD_T^\alpha\tilde v\right)\right|^{1/2}=|v|_{H_c^\alpha(0,T)}.$$

Here we also use the fact that $H_r^\alpha(\mathbf{R})=H_c^\alpha(\mathbf{R})$ and the norm $\|\cdot\|_{H_r^\alpha(\mathbf{R})}\cong\|\cdot\|_{H_c^\alpha(\mathbf{R})}$ which can be found in [11].

Next we prove that the norm $\|\cdot\|_{H_r^\alpha(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ in space $H_r^\alpha(0,T)=H_c^\alpha(0,T)$. In fact, following the ideas of the proof above, $\forall\, v\in H_r^\alpha(0,T)$, there exists a sequence $v_n\in C_0^\infty(0,T)$ such that

$$\|v_n-v\|_{H_r^\alpha(0,T)}\to 0,\quad n\to\infty,$$

and

$$\|v_n-v\|_{H_c^\alpha(0,T)}\to 0,\quad n\to\infty.$$

Thus

$$\begin{aligned}\|v\|_{H_r^\alpha(0,T)} &\le \|v-v_n\|_{H_r^\alpha(0,T)}+\|v_n\|_{H_r^\alpha(0,T)}\\ &\le \|v-v_n\|_{H_r^\alpha(0,T)}+C\|v_n\|_{H_c^\alpha(0,T)}\\ &\le \|v-v_n\|_{H_r^\alpha(0,T)}+C\|v_n-v\|_{H_c^\alpha(0,T)}+C\|v\|_{H_c^\alpha(0,T)}.\end{aligned}$$

Let $n\to\infty$, we get $\|v\|_{H_r^\alpha(0,T)}\le C\|v\|_{H_c^\alpha(0,T)}$ for any $v\in H_c^\alpha(0,T)$. Similarly we can prove $\|v\|_{H_c^\alpha(0,T)}\le C\|v\|_{H_r^\alpha(0,T)}$ for any $v\in H_r^\alpha(0,T)$. Thus the norm $\|\cdot\|_{H_r^\alpha(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ in $H_r^\alpha(0,T)=H_c^\alpha(0,T)$

Finally we turn to the proof of $H_c^\alpha(0,T)=H_0^\alpha(0,T)$ and the norm $\|\cdot\|_{H_c^\alpha(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_0^\alpha(0,T)}$ in $H_c^\alpha(0,T)=H_0^\alpha(0,T)$. The arguments of the proof are the same as the proof for $H_r^{\alpha}(0,T) = H_c^{\alpha}(0,T)$ above. Therefore it suffices to prove that the norm $\|\cdot\|_{H_c^{\alpha}(0,T)}$ is equivalent to the norm $\|\cdot\|_{H_0^{\alpha}(0,T)}$ in $C_0^{\infty}(0,T)$ which follows from

$$\begin{aligned} |v|_{H_c^{\alpha}(0,T)} &= |\tilde{v}|_{H_c^{\alpha}(\mathbf{R})} \cong |\tilde{v}|_{H_c^{\alpha}(\mathbf{R})} = \|\mathcal{F}\big({}_tD_T^{\alpha}\tilde{v}\big)\|_{L_2(\mathbf{R})} \\ &= \|(iw)^{\alpha}\mathcal{F}(\tilde{v})\|_{L_2(\mathbf{R})} \cong |\tilde{v}|_{H^{\alpha}(\mathbf{R})} = |v|_{H^{\alpha}(0,T)}. \end{aligned}$$

Here we used the Plancherel Theorem [26] and remark that $\mathcal{F}(\tilde{v})$ denotes the Fourier transform of $\tilde{v}$. □

Lemma **2.2**. [26] *We have*

(1) *If* $0 < p < 1,\ 0 < q < 1,\ v(0) = 0,\ t > 0,$ *then*

$${}_0D_t^{p+q}v(t) = \big({}_0D_t^{p}\big)\big({}_0D_t^{q}\big)v(t) = \big({}_0D_t^{q}\big)\big({}_0D_t^{p}\big)v(t), \quad \forall\, w \in H^{p+q}(0,T)$$

(2) *Let* $0 < \alpha < 1$. *Then we have*

$$\big({}_0D_t^{\alpha}w,\, v\big)_{L_2(0,T)} = \big(w,\, {}_tD_T^{\alpha}v\big)_{L_2(0,T)}, \quad \forall\, w \in H^{\alpha}(0,T),\ v \in C_0^{\infty}(0,T).$$

Lemma **2.3**. *Let* $0 < \alpha < 1$. *Then for any* $w \in {}_0H^{\alpha}(0,T)$, $v \in {}_0H^{\alpha/2}(0,T)$, *we have*

$$\big({}_0D_t^{\alpha}w,\, v\big)_{L_2(0,T)} = \big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v\big)_{L_2(0,T)}.$$

P r o o f. Since $0 < \alpha < 1$, we have [21]

$${}_0H^{\alpha/2}(0,T) = H_0^{\alpha/2}(0,T).$$

Thus, $\forall\, v \in {}_0H^{\alpha/2}(0,T)$, there exists a sequence $v_n \in C_0^{\infty}(0,T)$ such that

$$\|v - v_n\|_{H^{\alpha/2}(0,T)} \to 0, \quad n \to \infty.$$

By Lemma **2.2**, we have, for any $w \in {}_0H^{\alpha}(0,T)$ with $w(0) = 0$,

$$\begin{aligned} \big({}_0D_t^{\alpha}w,\, v_n\big)_{L_2(0,T)} &= \Big(\big({}_0D_t^{\alpha/2}\big)\big({}_0D_t^{\alpha/2}\big)w,\, v_n\Big)_{L_2(0,T)} \\ &= \big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v_n\big)_{L_2(0,T)}. \end{aligned}$$

We now prove that

$$\big({}_0D_t^{\alpha}w,\, v_n\big)_{L_2(0,T)} \to \big({}_0D_t^{\alpha}w,\, v\big)_{L_2(0,T)}, \quad n \to \infty, \tag{2.4}$$

$$\big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v_n\big)_{L_2(0,T)} \to \big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v\big)_{L_2(0,T)}, \quad n \to \infty, \tag{2.5}$$

It is easy to prove (2.4). For (2.5), we have

$$\begin{aligned}
&\Big|\big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v_n\big)_{L_2(0,T)} - \big({}_0D_t^{\alpha/2}w,\, {}_tD_T^{\alpha/2}v\big)_{L_2(0,T)}\Big| \\
&\leq \|{}_0D_t^{\alpha/2}w\|_{L_2(0,T)} \cdot \big\|{}_tD_T^{\alpha/2}v_n - {}_tD_T^{\alpha/2}v\big\|_{L_2(0,T)} \\
&\leq \|{}_0D_t^{\alpha/2}w\|_{L_2(0,T)} \cdot |v_n - v|_{H_r^{\alpha/2}(0,T)} \\
&\leq C\|{}_0D_t^{\alpha/2}w\|_{L_2(0,T)} \cdot \|v_n - v\|_{H^{\alpha/2}(0,T)} \to 0, \quad n \to \infty,
\end{aligned}$$

where we used Lemma **2.1** in the last inequality.

Together these estimates complete the proof of the Lemma **2.3**. □

Proof of Theorem **2.1**. The weak formulation of (1.1)-(1.3) is to find $u \in B^{\alpha/2}\big((0,T)\times\Omega\big)$ such that

$$\mathcal{A}(u,v) = \mathcal{F}(v), \quad \forall\, v \in B^{\alpha/2}\big((0,T)\times\Omega\big), \tag{2.6}$$

where the bilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{F}(v)$ are defined by using Lemma **2.3**,

$$\begin{aligned}
\mathcal{A}(u,v) &= \big({}_0D_t^{\alpha}u,\, v\big)_{L_2((0,T)\times\Omega)} + (\nabla u, \nabla v)_{L_2((0,T)\times\Omega)} \\
&= \big({}_0D_t^{\alpha/2}u,\, {}_tD_T^{\alpha/2}v\big)_{L_2((0,T)\times\Omega)} + (\nabla u, \nabla v)_{L_2((0,T)\times\Omega)},
\end{aligned}$$

and

$$\mathcal{F}(v) = (f, v)_{L_2((0,T)\times\Omega)}.$$

It is easy to prove the continuities of the bilinear form $\mathcal{A}(\cdot,\cdot)$ and the right hand functional $\mathcal{F}(v)$, that is, there exists a constant $C > 0$, such that

$$|\mathcal{A}(u,v)| \leq C\|u\|_{B^{\alpha/2}((0,T)\times\Omega)}\|v\|_{B^{\alpha/2}((0,T)\times\Omega)},$$

and

$$|\mathcal{F}(v)| \leq \|f\|_{L_2((0,T)\times\Omega)}\|v\|_{B^{\alpha/2}((0,T)\times\Omega)}. \tag{2.7}$$

We next prove the coercivity of the bilinear operator $\mathcal{A}(\cdot,\cdot)$ on $B^{\alpha/2}\big((0,T)\times\Omega\big)$. Note that [20]

$$\begin{aligned}
\big({}_0D_t^{\alpha/2}\varphi,\, {}_tD_T^{\alpha/2}\varphi\big)_{L_2(0,T)} &= \big({}_\infty D_t^{\alpha/2}\tilde{\varphi},\, {}_tD_\infty^{\alpha/2}\tilde{\varphi}\big)_{L_2(\mathbf{R})} \\
&= \cos\Big(\frac{\alpha}{2}\pi\Big) \cdot \|{}_\infty D_t^{\alpha/2}\tilde{\varphi}\|_{L_2(\mathbf{R})}, \quad \forall\, \varphi \in C_0^\infty(0,T),
\end{aligned}$$

where $\tilde{\varphi}$ is the extension of φ by zero outside of $(0,T)$. Thus we find that $\big({}_0D_t^{\alpha/2}v,\, {}_tD_T^{\alpha/2}v\big)_{L_2(0,T)}$ is nonnegative for $v \in H^{\alpha/2}(0,T)$, $0 < \alpha < 1$ since $\cos\big(\frac{\alpha}{2}\pi\big)$ is nonnegative for $0 < \alpha < 1$.

Combining this with Lemma **2.1**, we have

$$\begin{aligned}\mathcal{A}(v,v) &= \big({}_0D_t^{\alpha/2}v,\, {}_tD_T^{\alpha/2}v\big)_{L_2((0,T)\times\Omega)} + (\nabla v, \nabla v)_{L_2((0,T)\times\Omega)}\\ &\geq C\big({}_0D_t^{\alpha/2}v,\, {}_0D_t^{\alpha/2}v\big)_{L_2((0,T)\times\Omega)} + (\nabla v, \nabla v)_{L_2((0,T)\times\Omega)}\\ &\geq C\|v\|_{B^{\alpha/2}((0,T)\times\Omega)}, \end{aligned} \tag{2.8}$$

where we applied the Poincaré inequalities in the last inequality, that is

$$\|\nabla\varphi\|_{L_2(\Omega)} \cong \|\varphi\|_{H^1(\Omega)}, \quad \forall\, \varphi \in H_0^1(\Omega),$$

and

$$\big\|{}_0D_t^{\alpha/2}\psi\big\|_{L_2(0,T)} \cong \|\psi\|_{H^{\alpha/2}(0,T)}, \quad \forall\, \psi \in_0 H^{\alpha/2}(0,T).$$

By using the well-known Lax-Milgram Lemma, there exists a unique solution $u \in B^{\alpha/2}((0,T)\times\Omega)$ such that (2.6) holds.

To prove the stability estimate (2.1), we take $v = u$ in (2.6) to get, by using (2.8) and (2.7),

$$C\|u\|_{B^{\alpha/2}((0,T)\times\Omega)} \leq \mathcal{A}(u,u) = \mathcal{F}(u) \leq C\|f\|_{L_2((0,T)\times\Omega)}\|u\|_{B^{\alpha/2}((0,T)\times\Omega)},$$

which implies that

$$\|u\|_{B^{\alpha/2}((0,T)\times\Omega)} \leq C\|f\|_{L_2((0,T)\times\Omega)}.$$

The proof is complete. □

3. Time discretization

In this section, we will consider the time discretization of (1.1)- (1.3).

Define $A = -\Delta$, $\mathcal{D}(A) = H_0^1(\Omega)\cap H^2(\Omega)$. Then the system (1.1)-(1.3) can be written in the abstract form

$${}_0^RD_t^{\alpha}u(t) + Au(t) = f(t), \quad 0 < t < T,\ 0 < \alpha < 1, \tag{3.1}$$

$$u(0) = u_0, \tag{3.2}$$

or, equivalently,

$${}_0^RD_t^{\alpha}[u-u_0](t) + Au(t) = f(t), \quad 0 < t < T,\ 0 < \alpha < 1. \tag{3.3}$$

Note that

$${}_0^RD_t^{\alpha}u(t) = \frac{1}{\Gamma(-\alpha)}\frac{d}{dt}\int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha}}\,d\tau = \frac{1}{\Gamma(-\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{\alpha+1}}\,d\tau,$$

where the integral must be interpreted as a Hadamand finite-part integral [6].

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$. Then, for fixed $t_j, j = 1,2,\dots n$, we have

$$
\begin{aligned}
{}_0^R D_t^\alpha [u-u_0](t_j) &= \frac{1}{\Gamma(-\alpha)} \int_0^{t_j} \frac{u(\tau)-u(0)}{(t-\tau)^{1+\alpha}}\, d\tau \\
&= \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 \frac{u(t_j - t_j w) - u(0)}{w^{1+\alpha}}\, dw = \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \int_0^1 g(w) w^{-1-\alpha}\, dw,
\end{aligned}
$$

where $g(w) = u(t_j - t_j w) - u(0)$.

Now, for every j, we replace the integral by a first-degree compound quadrature formula with the equispaced nodes $0, \frac{1}{j}, \frac{2}{j}, \dots, \frac{j}{j}$ and obtain

$$
\int_0^1 g(w) w^{-1-\alpha}\, dw = \sum_{k=0}^{j} \alpha_{kj} g(k/j) + R_j(g),
$$

where the weights α_{kj} satisfy that [6]

$$
\alpha(1-\alpha) j^{-\alpha} \alpha_{kj} = \begin{cases} -1, & \text{for } k = 0, \\ 2k^{1-\alpha} - (k-1)^{1-\alpha} - (k+1)^{1-\alpha}, & \text{for } k = 1, 2, \dots, j-1, \\ (\alpha - 1) k^{-\alpha} - (k-1)^{1-\alpha} + k^{1-\alpha}, & \text{for } k = j, \end{cases}
$$

and the remainder term $R_j(g)$ satisfies

$$
\|R_j(g)\| \le C j^{\alpha - 2} \sup_{0 \le t \le T} \|g''(t)\|.
$$

Thus we have

$$
\begin{aligned}
{}_0^R D_t^\alpha [u-u_0](t_j) &= \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} \Big(\sum_{k=0}^{j} \alpha_{kj} \big(u(t_j - t_k) - u(0) \big) + R_j(g) \Big) \\
&= \Delta t^{-\alpha} \sum_{k=0}^{j} w_{kj} \big(u(t_j - t_k) - u(0) \big) + \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g),
\end{aligned}
$$

where

$$
\Gamma(2-\alpha) w_{kj} = \begin{cases} 1, & \text{for } k = 0, \\ -2k^{1-\alpha} + (k-1)^{1-\alpha} + (k+1)^{1-\alpha}, & \text{for } k = 1, 2, \dots, j-1, \\ -(\alpha - 1) k^{-\alpha} + (k-1)^{1-\alpha} - k^{1-\alpha}, & \text{for } k = j. \end{cases}
$$

Let $t = t_j$. We can write (3.3) as

$$
\Delta t^{-\alpha} \sum_{k=0}^{j} w_{kj} \big(u(t_j - t_k) - u(0) \big) + A u(t_j) = f(t_j) - \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g), \; j = 1, 2, 3, \dots. \tag{3.4}
$$

Denote $U^j \approx u(t_j)$ as the approximation of $u(t_j)$. We can define the following time stepping method

$$\Delta t^{-\alpha} \sum_{k=0}^{j} \big(U^{j-k} - U^0\big) + AU^j = f_j, \quad f_j = f(t_j). \tag{3.5}$$

Let $e^j = U^j - u(t_j)$ denote the error. Then we have the following error estimate:

THEOREM **3.1**. *Let U^n and $u(t_n)$ be the solutions of* (3.5) *and* (3.1), *respectively. Then we have*

$$\|U^n - u(t_n)\| \leq C\Delta t^{2-\alpha}.$$

In order to prove this Theorem, we need the following Lemma.

LEMMA **3.1**. [6] *For $0 < \alpha < 1$, let the sequence $\{d_j\}\, j = 1, 2, \ldots$ be given by $d_1 = 1$ and*

$$d_j = 1 + \alpha(1-\alpha) j^{-\alpha} \sum_{k=1}^{j-1} \alpha_{kj} d_{j-k}, \; j = 2, 3, \cdots,$$

where α_{kj} is as in (3.4). *Then,*

$$1 \leq d_j \leq \frac{\sin \pi\alpha}{\pi\alpha(1-\alpha)} j^{\alpha}, \; j = 1, 2, \ldots.$$

Proof of Theorem **3.1**. Subtracting (3.5) from (3.4), we get the error equation

$$\Delta t^{-\alpha} \sum_{k=0}^{j} w_{kj} \big(e^{j-k} - e^0\big) + Ae^j = -\frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g),$$

or

$$\begin{aligned} e^j &= \big(\Delta t^{-\alpha} w_{0j} + A\big)^{-1} \Big(\Delta t^{-\alpha} \sum_{k=1}^{j} w_{kj} e_{j-k} - \frac{t_j^{-\alpha}}{\Gamma(-\alpha)} R_j(g)\Big) \\ &= (\alpha_{0j} + \Gamma(-\alpha) t_j^{\alpha} A)^{-1} \Big(\sum_{k=1}^{j} \alpha_{kj} e_{j-k} - R_j(g)\Big). \end{aligned}$$

Thus,

$$\|e^j\| \leq \big\|(\alpha_{0j} + \Gamma(-\alpha) t_j^{\alpha} A)^{-1}\big\| \Big(\Big(\sum_{k=1}^{j} \alpha_{kj} \|e_{j-k}\| + \|R_j(g)\|\Big).$$

Note that A is a positive definite elliptic operator, we have, since $\alpha_{0j} < 0$ and $\Gamma(-\alpha) < 0$,

$$\left\|(\alpha_{0j} + \Gamma(-\alpha)t_j^\alpha A)^{-1}\right\| = \sup_{\lambda>0}\left|(\alpha_{0j} + \Gamma(-\alpha)t_j^\alpha \lambda)^{-1}\right| < -\alpha_{0j}^{-1}.$$

Hence

$$\begin{aligned}
\|e^j\| &\leq -\alpha_{0j}^{-1}\Big(\sum_{k=1}^{j}\alpha_{kj}\|e_{j-k}\| + Cj^{\alpha-2}n^{-2}\sup_{0\leq t\leq T}\|u''(t)\|\Big) \\
&= \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j}\alpha_{kj}\|e_{j-k}\| + \alpha(1-\alpha)Cn^{-2}\sup_{0\leq t\leq T}\|u''(t)\|.
\end{aligned}$$

Note that $e^0 = u(0) - U^0 = 0$. Denote

$$\begin{aligned}
d_1 &= 1, \\
d_j &= 1 + \alpha(1-\alpha)j^{-\alpha}\sum_{k=1}^{j-1}\alpha_{kj}d_{j-k}, \quad j = 2, 3, \ldots, n.
\end{aligned}$$

Then we have by induction,

$$\|e^j\| \leq C\alpha(1-\alpha)n^{-2}\sup_{0\leq t\leq T}\|u''(t)\|\cdot d_j.$$

By Lemma **3.1**, we get

$$\|e^j\| \leq C\frac{\sin\pi\alpha}{\pi}\sup_{0\leq t\leq T}\|u''(t)\| j^{-\alpha}n^{-2} \leq C\Delta t^{2-\alpha}.$$

The proof is complete.

□

4. Space discretization

In this section, we will consider the space discretization of (1.1) - (1.3).

The variational form of (1.1) - (1.3) is to find $u(t) \in H_0^1(\Omega)$, such that,

$$\big({}_0D_t^\alpha u(t), v)_{L_2(\Omega)} + (\nabla u(t), \nabla v)_{L_2(\Omega)} = (f(t), v)_{L_2(\Omega)}, \quad \forall\, v \in H_0^1(\Omega). \tag{4.1}$$

Let $\mathcal{T}$ denote a partition of Ω into disjoint triangles such that no vertex of any triangle lies on the interior of a side of another triangle and such that the union of the triangles determines a polygonal domain $\Omega_h \subset \Omega$ with boundary vertices on $\partial\Omega$. Let h denote the maximal length of the sides of the triangulation $\mathcal{T}_h$. We assume that the triangulations are quasiuniform in the sense that the triangles of $\mathcal{T}_h$ are of essentially the same size.

Let r be any nonnegative integer. We denote by $\|\cdot\|_{H^r(\Omega)}$ the norm in $H^r(\Omega)$. Let $S_h \subset H_0^1$ be a family of finite element space with the accuracy of order $r \geq 2$, i.e., S_h consists of continuous functions on the closure $\bar{\Omega}$ of Ω which are polynomials of degree at most $r-1$ in each triangle of $\mathcal{T}_h$ and which vanish outside Ω_h, such that, for small h, with $v \in H^s(\Omega) \cap H_0^1(\Omega)$ [33]

$$\inf_{\chi \in S_h} \|v-\chi\|_{L_2(\Omega)} + h\|\nabla(v-\chi)\|_{L_2(\Omega)} \leq Ch^s \|v\|_{L_2(\Omega)}, \quad \text{for } 1 \leq s \leq r. \tag{4.2}$$

The semidiscrete problem of (4.1) is to find the approximate solution $u_h(t) = u_h(\cdot, t) \in S_h$ for each t such that

$$\left({}_0D_t^\alpha u_h(t), \chi\right)_{L_2(\Omega)} + (\nabla u_h(t), \nabla \chi)_{L_2(\Omega)} = (f(t), \chi)_{L_2(\Omega)}, \quad \forall\, \chi \in S_h. \tag{4.3}$$

Let $R_h : H_0^1(\Omega) \to S_h$ be the elliptic projection, or Ritz projection, defined by

$$(\nabla(R_h u), \nabla \chi) = (\nabla u, \nabla \chi), \quad \forall\, \chi \in S_h. \tag{4.4}$$

We then have,

Lemma **4.1**. [33] *Assume that* (4.2) *holds. Then, with R_h defined by* (4.4) *we have, with $v \in H^s(\Omega) \cap H_0^1(\Omega)$,*

$$\|R_h v - v\|_{L_2(\Omega)} + h\|\nabla(R_h v - v)\|_{L_2(\Omega)} \leq Ch^s \|v\|_{H^s(\Omega)}, \quad \text{for } 1 \leq s \leq r.$$

Lemma **4.2**. *Let $0 < \alpha < 1$ and assume that $w \in H^\alpha((0,T), L_2(\Omega))$. We have*

$$\begin{aligned} \int_0^T \left({}_0D_t^\alpha w(t), w(t)\right)_{L_2(\Omega)} dt &= \int_0^T \left({}_0D_t^{\alpha/2} w(t),\, {}_tD_T^{\alpha/2} w(t)\right)_{L_2(\Omega)} dt \\ &= \int_0^T \left({}_0D_t^{\alpha/2} w(t),\, {}_0D_t^{\alpha/2} w(t)\right)_{L_2(\Omega)} dt \end{aligned}$$

P r o o f. We have, by Lemmas **2.3**, **2.1**,

$$\int_0^T \left({}_0D_t^\alpha w(t), w(t)\right)_{L_2(\Omega)} dt \tag{4.5}$$

$$= \int_0^T \int_\Omega \left({}_0D_t^\alpha w(x,t)\right) w(x,t)\, dx dt$$

$$= \int_\Omega \left({}_0D_t^\alpha w(x,\cdot), w(x,\cdot)\right)_{L_2(0,T)} dx \tag{4.6}$$

$$= \int_\Omega \left({}_0D_t^{\alpha/2} w(x,\cdot),\, {}_tD_T^{\alpha/2} w(x,\cdot)\right)_{L_2(0,T)} dx$$

$$= \int_0^T \left({}_0D_t^{\alpha/2} w(t),\, {}_tD_T^{\alpha/2} w(t)\right)_{L_2(\Omega)} dt.$$

Similarly, we can prove

$$\int_0^T \big({}_0D_t^{\alpha} w(t), w(t)\big)_{L_2(\Omega)}\, dt = \int_0^T \big({}_0D_t^{\alpha/2} w(t),\, {}_0D_t^{\alpha/2} w(t)\big)_{L_2(\Omega)}\, dt.$$

The proof is complete.

□

We now come to the main theorem in this section.

THEOREM **4.1**. *Let u_h and u be the solutions of* (4.1) *and* (4.3). *Then*

$$\int_0^T \|{}_0D_t^{\alpha/2}(u_h(t) - u(t))\|^2_{L_2(\Omega)}\, dt \le Ch^{2r} \int_0^T \|{}_0D_t^{\alpha/2} u(t)\|^2_{H^r(\Omega)}\, dt.$$

P r o o f. We write

$$u_h - u = \theta + \rho, \quad \text{where } \theta = u_h - R_h u,\ \rho = R_h u - u.$$

The second term is easily bounded by Lemma **3.1** and has the obvious estimates

$$\int_0^T \|{}_0D_t^{\alpha/2}\rho(t)\|^2_{L_2(\Omega)}\, dt \le Ch^{2r} \int_0^T \|{}_0D_t^{\alpha/2} u(t)\|^2_{H^r(\Omega)}\, dt. \tag{4.7}$$

In order to estimate θ, we note that by our definitions,

$$\begin{aligned}
&({}_0D_t^{\alpha}\theta(t), \chi)_{L_2(\Omega)} + (\nabla\theta(t), \nabla\chi)_{L_2(\Omega)} \\
&= ({}_0D_t^{\alpha}(u_h(t) - R_h u(t)), \chi)_{L_2(\Omega)} + (\nabla(u_h(t) - R_h u(t)), \nabla\chi)_{L_2(\Omega)} \\
&= ({}_0D_t^{\alpha} u_h(t), \chi)_{L_2(\Omega)} + (\nabla u_h(t), \nabla\chi)_{L_2(\Omega)} - ({}_0D_t^{\alpha} R_h u(t), \chi)_{L_2(\Omega)} + (\nabla R_h u(t), \nabla\chi)_{L_2(\Omega)} \\
&= (f(t), \chi)_{L_2(\Omega)} - ({}_0D_t^{\alpha} R_h u(t), \chi)_{L_2(\Omega)} + (\nabla R_h u(t), \nabla\chi)_{L_2(\Omega)} \\
&= ({}_0D_t^{\alpha} u(t), \chi)_{L_2(\Omega)} - ({}_0D_t^{\alpha} R_h u(t), \chi)_{L_2(\Omega)} \\
&= ((I - R_h){}_0D_t^{\alpha} R_h u(t), \chi)_{L_2(\Omega)} = ({}_0D_t^{\alpha}\rho(t), \chi)_{L_2(\Omega)}, \quad \forall\, \chi \in S_h.
\end{aligned}$$

Choose $\chi = \theta(t)$ and integrating on both sides with respect to t on $[0, T]$, we obtain

$$\int_0^T \big({}_0D_t^{\alpha}\theta(t), \theta(t)\big)_{L_2(\Omega)}\, dt + \int_0^T \big(\nabla\theta(t), \nabla\chi\big)_{L_2(\Omega)}\, dt = \int_0^T \big({}_0D_t^{\alpha}\rho(t), \theta(t)\big)_{L_2(\Omega)}\, dt.$$

By Lemma **4.2**, we have, for any small $\epsilon > 0$,

$$\int_0^T \|{}_0D_t^{\alpha/2}\theta(t)\|_{L_2(\Omega)}^2\,dt + \int_0^T \|\nabla\theta(t)\|_{L_2(\Omega)}^2\,dt$$
$$= \int_0^T ({}_0D_t^{\alpha}\rho(t), \theta(t))_{L_2(\Omega)}\,dt = \int_0^T ({}_0D_t^{\alpha/2}\rho(t),\, {}_0D_t^{\alpha/2}\theta(t))_{L_2(\Omega)}\,dt$$
$$\leq \int_0^T \Big(C_\epsilon \|{}_0D_t^{\alpha/2}\rho(t)\|_{L_2(\Omega)}^2\,dt + \epsilon \|{}_0D_t^{\alpha/2}\theta(t)\|_{L_2(\Omega)}^2\Big)\,dt.$$

For sufficiently small $\epsilon > 0$, we get, by (4.7)

$$\int_0^T \|{}_0D_t^{\alpha/2}\theta(t)\|_{L_2(\Omega)}^2\,dt + \int_0^T \|\nabla\theta(t)\|_{L_2(\Omega)}^2\,dt$$
$$\leq C\int_0^T \|{}_0D_t^{\alpha/2}\rho(t)\|_{L_2(\Omega)}^2\,dt \leq Ch^{2r}\int_0^T \|{}_0D_t^{\alpha/2}u(t)\|_{H^r(\Omega)}^2\,dt. \tag{4.8}$$

Combining (4.7) with (4.8), we complete the proof of the theorem.

□

COROLLARY **4.1**. *Let u_h and u be the solutions of* (4.1) *and* (4.3). *Then*

$$\int_0^T \|u_h(t) - u(t)\|_{L_2(\Omega)}^2\,dt \leq Ch^{2r}\int_0^T \|{}_0D_t^{\alpha/2}u(t)\|_{H^r(\Omega)}^2\,dt. \tag{4.9}$$

P r o o f. Note that, by Theorem **4.1**,
We have

$$\int_0^T \|u_h(t) - u(t)\|_{L_2(\Omega)}^2\,dt = \int_0^T\int_\Omega |u_h(x,t) - u(x,t)|^2\,dxdt$$
$$= \int_\Omega\int_0^T |u_h(x,t) - u(x,t)|^2\,dtdx \leq \int_\Omega\int_0^T |{}_0D_t^{\alpha}(u_h(x,t) - u(x,t))|^2\,dtdx$$
$$\leq \int_0^T \|{}_0D_t^{\alpha}(u_h(t) - u(t)\|_{L_2(\Omega)}^2\,dt \leq Ch^{2r}\int_0^T \|{}_0D_t^{\alpha/2}u(t)\|_{H^r(\Omega)}^2\,dt,$$

which is (4.9). The proof of the Lemma is complete. □

5. Numerical simulations

In this section, we present some numerical results by using the finite element method for solving the time-fractional partial differential equation (1.1) – (1.3). The numerical results are consistent with our theoretical results. We can see that convergence rate of numerical solutions is of order

$2-\alpha$ as the time stepsize tends to zero and of order 2 as the space step size tends to zero, on condition that the exact solution is smooth.

EXAMPLE **5.1**. Consider

$$ {}_0^R D_t^{\alpha} u(x,t) - \frac{d^2}{dx^2} u(x,t) = f(x,t), \quad t \in [0,T],\ 0 < x < 1, \tag{5.1} $$

$$ u(x,0) = 0, \quad 0 < x < 1, \tag{5.2} $$

$$ u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{5.3} $$

where

$$ f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \sin(2\pi x) + 4\pi^2 \sin(2\pi x) t^2. $$

The exact solution is $u(x,t) = t^2 \sin 2\pi x$.

The main purpose of these experiments is to check the convergence rate of the numerical solutions with respect to the fractional order α. We use the linear finite element method and therefore the convergence order is $O(\Delta t^{2-\alpha} + \Delta x^2)$.

In the first test, we fix $T = 1$, $\alpha = 0.5$ and $\Delta x = 0.001$ which is small enough such that the space discretization errors are negligible as compared with the time errors. We choose stepsize $\Delta t = 1/2^i$ $(i = 1, \cdots, 5)$, then we obtain Table 1 with the estimated convergence rate when $\alpha = 0.5$, tending to a limit close to 1.5. In the same way, we can plot the errors in the logscale as functions of the $\log(\Delta t^{-1})$ for $\alpha = 0.2, 0.5, 0.9$ in Figures 1, 2, 3 and obtain the convergence rates. Here we investigate both the L^2-norm and the H^1-norm in space.

Δx	Δt	H^1-norm	2-norm	estimated cvgce. rates
0.001	0.5000	0.01822017	0.00250060	
0.001	0.2500	0.00675406	0.00092695	1.4317
0.001	0.1250	0.00245740	0.00033726	1.4586
0.001	0.0625	0.00087822	0.00012053	1.4845
0.001	0.03125	0.00030493	4.18492564e-05	1.5261

TABLE 1. Example 1, Fix $\alpha = 0.5$ $\Delta x = 0.001$

In Figures 1-3 , one can observe that the error curves are all nearly straight lines. The convergence orders are the slopes of the lines respectively. By example 1, we can observe that the convergence order of the method with respect to the time step is $2-\alpha$, which have been shown in Table 2.

α	0.1	0.2	0.3	0.4	0.5
est. cvgce. rate	1.8805	1.8343	1.6786	1.5712	1.4752
α	0.6	0.7	0.8	0.9	
est. cvgce. rate	1.3818	1.2882	1.1935	1.0976	

TABLE 2. Convergence rates in Example 1

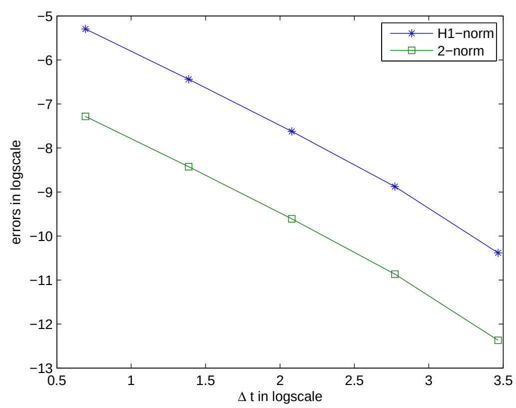

FIGURE 1. Example 1, H^1 norm and L^2 norm of errors with $\alpha = 0.2$ $\Delta x = 0.001$

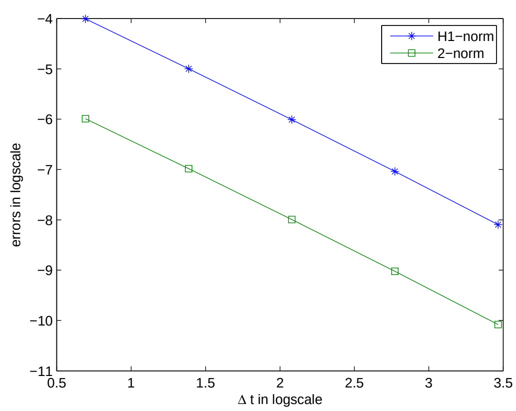

FIGURE 2. Example 1, H^1 norm and L^2 norm of errors with $\alpha = 0.5$ $\Delta x = 0.001$

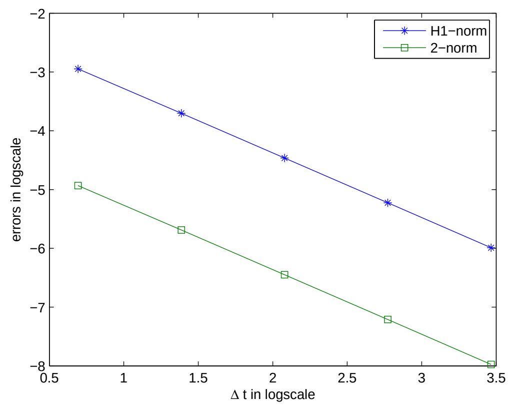

FIGURE 3. Example 1, H^1 norm and L^2 norm of errors with $\alpha = 0.9$ $\Delta x = 0.001$

On the other hand, if we fix Δt small enough, then the convergence rate of space discretization errors can be shown clearly (see Table 3 and figure 4). In this case $\alpha = 0.5$, the limiting vale of the convergence rate is consistent with the value of 2 that is expected from the theory.

Δx	Δt	H^1-norm	2-norm	est. cvge. rates
0.2500	0.001	0.79733222	0.10885742	
0.1250	0.001	0.23998057	0.03202152	1.7653
0.0625	0.001	0.06266713	0.00842864	1.9257
0.03125	0.001	0.01584965	0.00215046	1.9707
0.015625	0.001	0.00397465	0.00054228	1.9875

TABLE 3. Example 1, Fix $\alpha = 0.5$ $\Delta t = 0.001$

EXAMPLE **5.2**. Consider

$$ {}_0^R D_t^{\alpha} u(x,t) - \frac{d^2}{dx^2} u(x,t) = f(x,t), \quad t \in [0,T],\ 0 < x < 1, \tag{5.4} $$

$$ u(x,0) = 0, \quad x \in \Omega, \tag{5.5} $$

$$ u(0,t) = u(1,t) = 0, \quad t \in [0,T]. \tag{5.6} $$

The exact solution is $u(x,t) = \sin \pi t \sin \pi x$.

In this example, we fix $T = 1$, $\alpha = 0.5$ and $\Delta x = 0.0002$. We obtain the convergence rate in Table 4. In Figure 5, we plot the errors in logscale

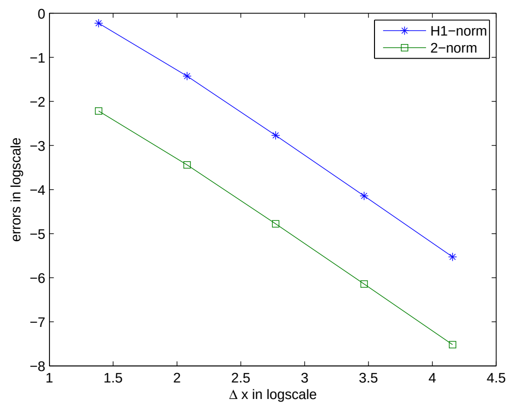

FIGURE 4. Example 1, H^1 norm and L^2 norm of errors with $\alpha = 0.5$ $\Delta t = 0.001$

as functions of time stepsize $\log(\Delta t^{-1})$ for $\alpha = 0.5$. Here we can observe much better convergence than predicted by the theory, and this is worthy of further investigation.

Δx	Δt	H^1-norm	2-norm	est. cvge. rates
0.0002	0.5000	0.10734868	0.02591770	
0.0002	0.2500	0.02757097	0.00665659	1.9611
0.0002	0.1250	0.00541156	0.00130654	2.3490
0.0002	0.0625	0.00057297	0.00013833	3.2395

TABLE 4. Example 2, Fix $\alpha = 0.5$ $\Delta x = 0.002$.

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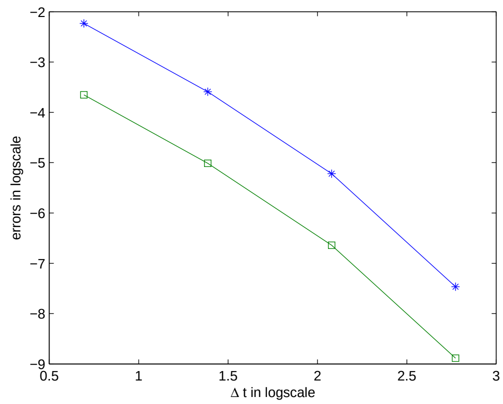

FIGURE 5. Example 2, H^1 norm and L^2 norm of errors with $\alpha = 0.5$ $\Delta x = 0.0002$

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